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Institute

Quantum field theory meets higher categories

Claudia Scheimbauer

Mathematical Institute
University of Oxford

Swiss National Science Foundation



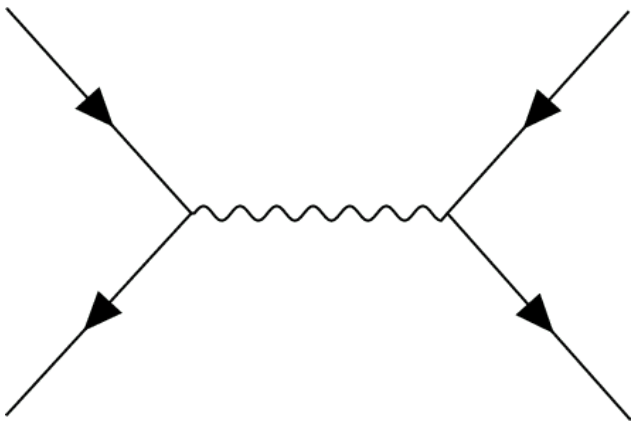
Oxford
Mathematics

Quantum field theory meets higher categories

... or topology?

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Quantum field theory



Topology

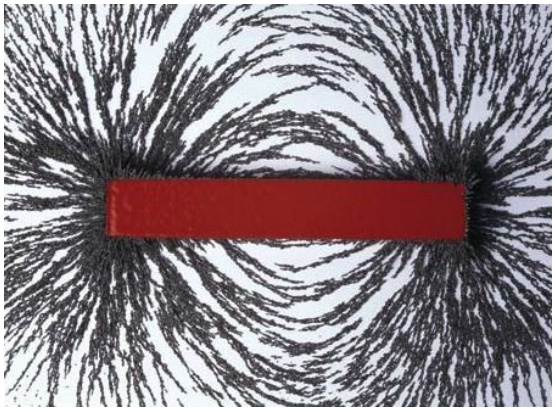


image: Sebastian Huber, ETH Zurich

Field theory

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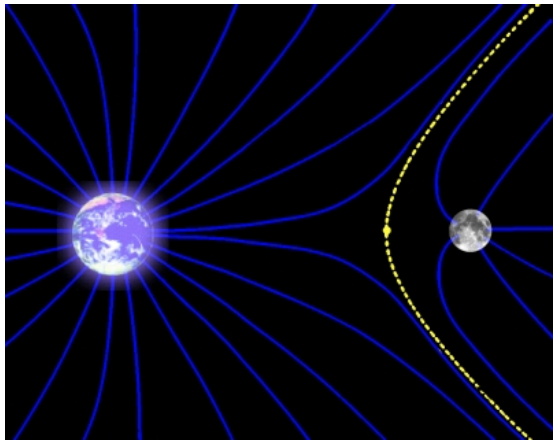
...classically, study *fields*, such as electro-magnetic, gravitational fields



images:

kullabs.com,

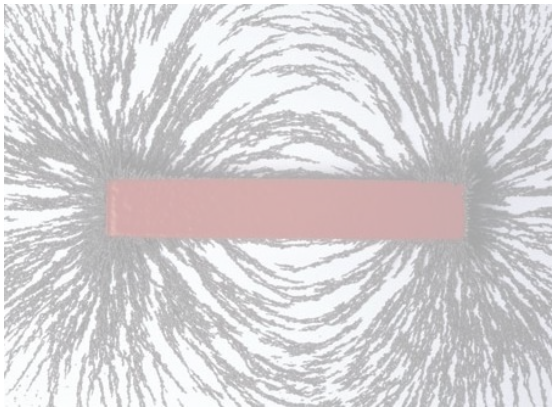
xkcd.com

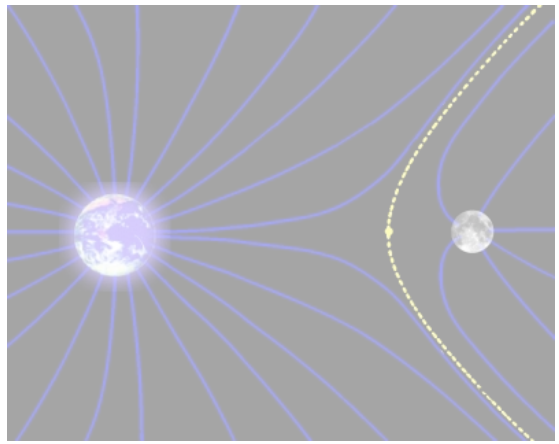


Field theory

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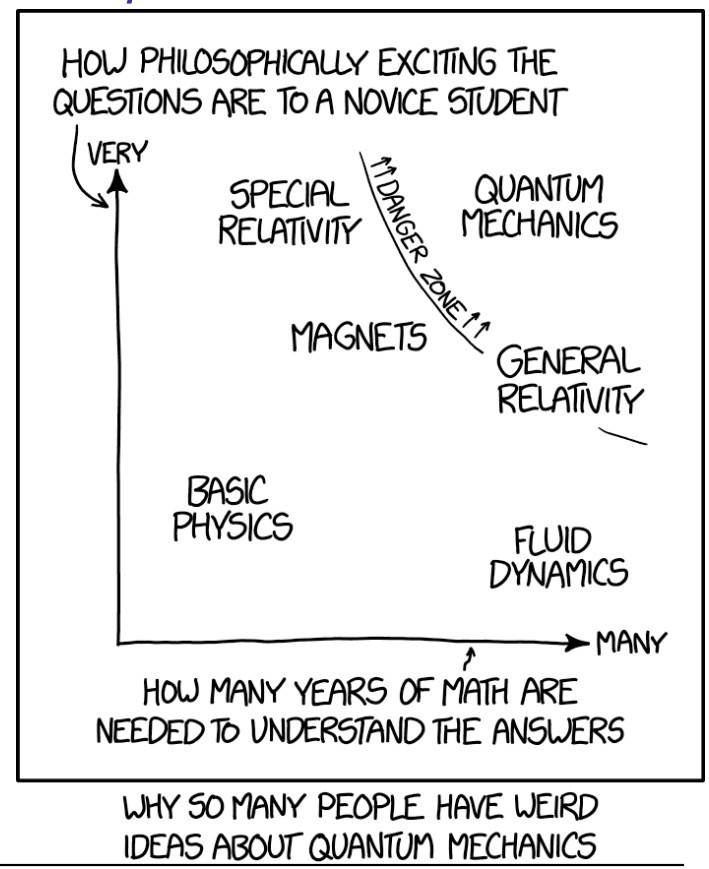


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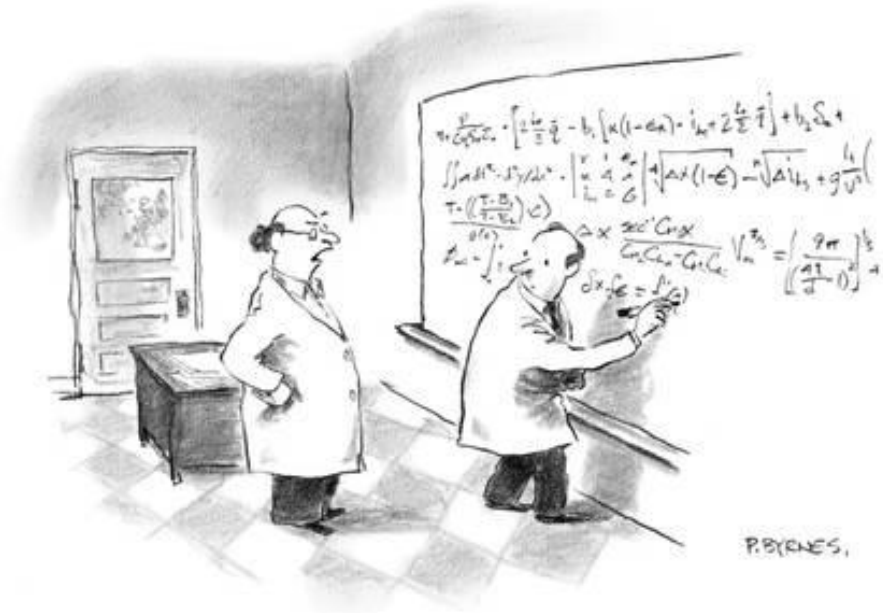
kullabs.com,

xkcd.com

...*quantum* fields?



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"Duh!"

One of the guiding principles is the goal of understanding the ultimate foundations of physics. [...] But, [...] it is also true that getting there is more than half the fun: If a physical insight leads to a significant new result in mathematics, that is considered a success. – *G. Moore*

Topology

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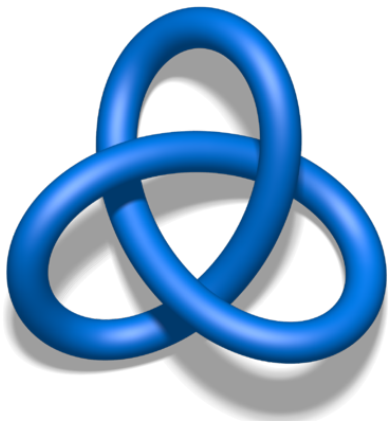
...study *spaces*, up to deformation ("homeomorphism")



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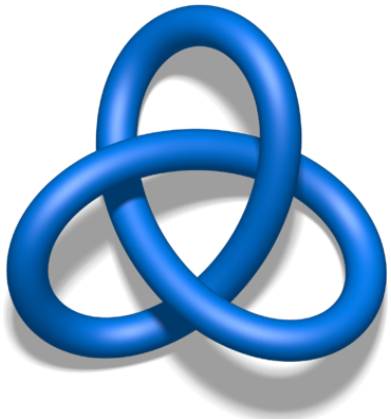




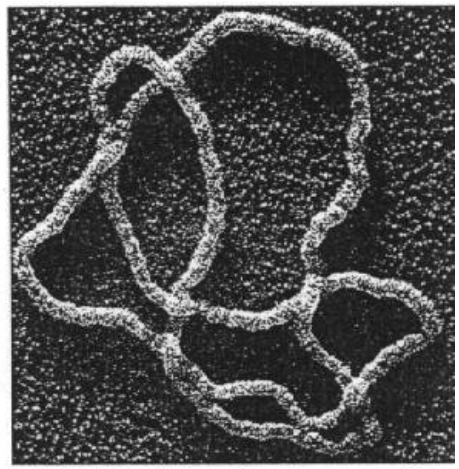
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...study *spaces*, up to deformation ("homeomorphism")









Category

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... a structure/language useful to organizing data:

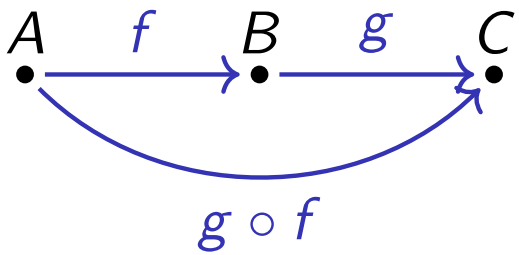
- objects

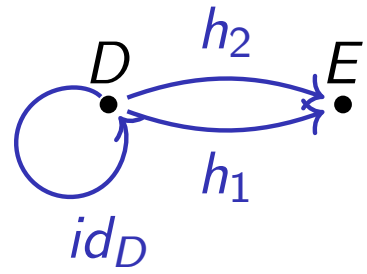


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... a structure/language useful to organizing data:

- objects
- *morphisms*

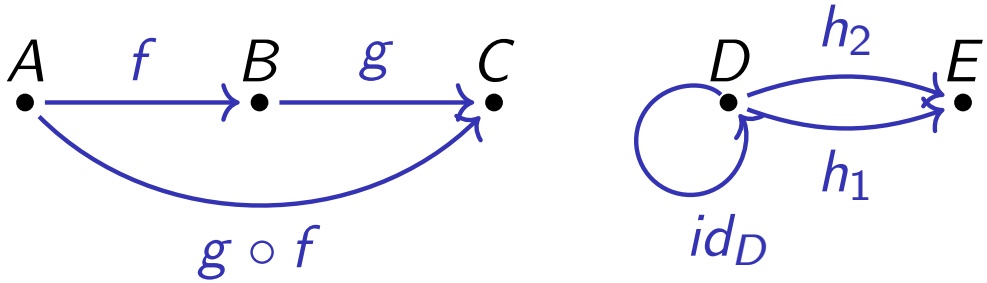




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... a structure/language useful to organizing data:

- objects
- *morphisms*



category	objects	*morphisms*
Set	sets	*functions*
Group	groups	*group homomorphisms*
Mfld_n	nd smooth manifolds	*smooth maps*
$\mathrm{Vect}_{\mathbb{C}}$	$\mathbb{C}$-vector spaces	*linear maps*
Alg^{hom}	$\mathbb{C}$-algebras	*algebra homomorphisms*
Alg	$\mathbb{C}$-algebras	*(A, B)-bimodules*

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Definition (1980's, Atiyah, Segal; after ideas by Witten)

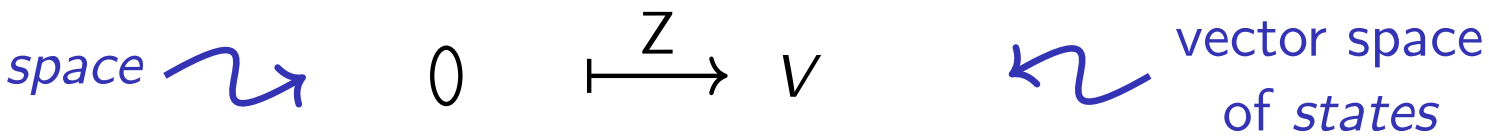
A *2-dimensional topological field theory* is an assignment Z as follows:



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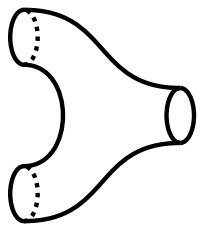


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Definition (1980's, Atiyah, Segal; after ideas by Witten)

A *2-dimensional topological field theory* is an assignment Z as follows:

space ↝ $0 \overset{Z}{\longmapsto} V$ ↜ vector space of *states*

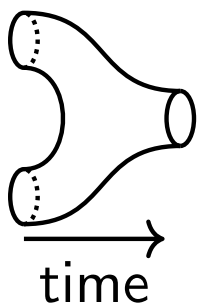
spacetime ↝ 

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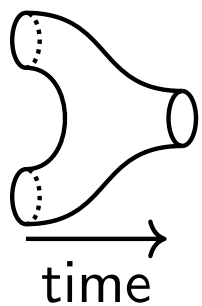
$\overset{Z}{\longmapsto} (V \otimes V \to V)$ ↜ *evolution* operator

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space $0 \xrightarrow{Z} V$ vector space of *states*

spacetime 

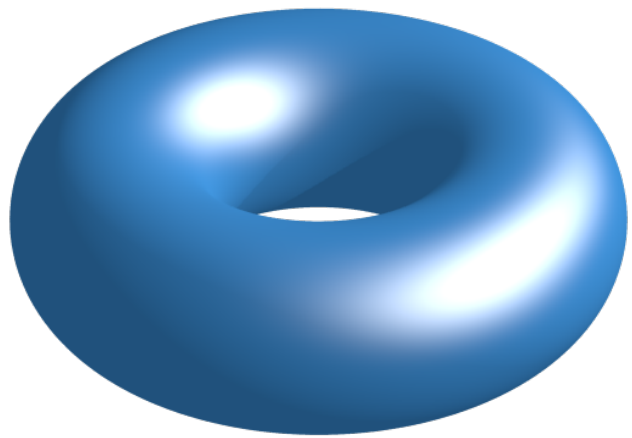
$\xrightarrow{Z} (V \otimes V \to V)$ *evolution* operator

$$\text{Spacetimes} \xrightarrow{Z} \text{Vect}_{\mathbb{C}}$$

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extending TFTs

- using "higher categories"
- incorporating "locality"



An extended topological field theory is...

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Spacetimes $\longrightarrow$ Vect

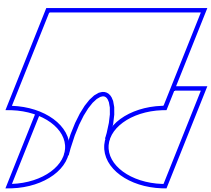
 $\longmapsto$ linear map

• • $\longmapsto$ vector space

An extended topological field theory is...

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Spacetimes $\longrightarrow$ Vect ?


$\longmapsto$

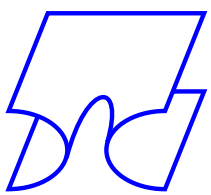

$\longmapsto$ linear map ?

• • $\longmapsto$ vector space

An extended topological field theory is...

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Spacetimes $\longrightarrow$ Vect $2\text{Vect} \overset{?}{=} \text{Alg}_1$


$\longmapsto$ homomorphism

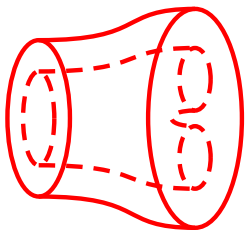

$\longmapsto$ linear map bimodule ?

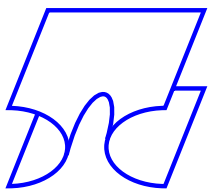
• • $\longmapsto$ vector space algebra

An extended topological field theory is...

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Spacetimes $\longrightarrow$ Vect 2Vect $\stackrel{?}{=}$ Alg_1 ?


$\longmapsto$


$\longmapsto$ homomorphism

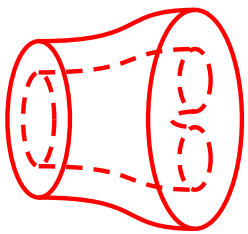
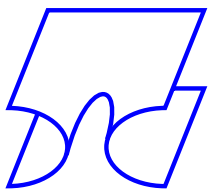

$\longmapsto$ linear map bimodule ?

?

• • $\longmapsto$ vector space algebra

An extended topological field theory is...

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Spacetimes $\longrightarrow$	Vect	2Vect $\overset{?}{=}$ Alg_1	3Vect $\overset{?}{=}$ Alg_2
 $\longmapsto$			homomorphism
 $\longmapsto$		homomorphism	bimodule of bimodules
 $\longmapsto$	linear map	bimodule $\overset{?}{}$	bimodule $\overset{?}{}$
 $\longmapsto$	vector space	algebra	E_2-algebra

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Theorem (Baez-Dolan, Lurie, Schommer-Pries, Ayala-Francis,...)

- *An extended n-dimensional TFT is fully determined by its value at a point.*
- *Conversely, any "n-dualizable" object is the value of a point of an extended n-dimensional TFT.*

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Example ($n = 1$)

A vector space V is 1-dualizable in $\mathrm{Vect}_{\mathbb{C}}$ iff there is

- a dual $V^* = \mathrm{Hom}(V, \mathbb{C})$,
- an evaluation $ev : V^* \otimes V \to \mathbb{C}$,
- a coevaluation $coev : \mathbb{C} \to V \otimes V^*$

such that (...)

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such that (...) $\Leftrightarrow (V^*)^* \cong V$ *(finite dimensional)*

Applications and outlook

...or, what do I (try to) do?

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- Construct examples!
  - without using Cobordism Hypothesis (explicitly, geometric)
  - with CH (check n-dualizability, algebraic)
- Extend framework to allow for more examples
  (twisted, anomalous versions)
- Develop framework/language for generalizations
- Incorporate more physics... more geometric structures?